A comparison of some Popular Mathematical Models used for the determination of the Physical Production Function in respect of Univariate Fertilizer Responses

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THE determination of economic fertilizer dosages is necessarily the ultimate goal in any fertilizer experimental programme on agricultural crops; and rightly so the agricultural scientist is now increasingly conscious of the importance of the economics of fertilizer use. However as the major portion of the techniques involved in the evaluation of the economic dosages of fertilizers—specially the determination of the physical production function relating fertilizer inputs to yield—falls within the domain of the Statistician or the Agricultural Economist, it is felt that any hints of guidance in such techniques will be welcome by most workers engaged in this field of research. In fact, even if the services of a Statistician were available, such information may be useful in that it will enable the Research worker to satisfy himself independently that the mathematical equation fitted by the Statistician does not run contrary to scientific principles.

This paper is devoted to a comparison of some popular mathematical models used for fitting curves on univariate fertilizer responses (namely, responses to different levels of a single nutrient or of a predetermined fertilizer mixture), with a view to determining a more universally applicable model. It also deals with a background review of some basic considerations regarding fertilizer response patterns as it is felt that such a prelude will be useful to a better appreciation of the above comparisons.

1. A REVIEW OF SOME POPULAR NOTIONS REGARDING FERTILIZER RESPONSE PATTERNS

(a) *The generalized fertilizer response curve*

Ever since the classical work of Mitscherlich (1909) on fertilizer responses, several investigators, on the basis of experiments with the effects on yields of varying fertilizer inputs, have been led to conclude that the production function relating fertilizer inputs to yield is intrinsically an ' S ' shaped curve. Jacob & Uexküll (1960) in describing this trend observe that " very small dressings of fertilizers have mostly no effect at all or at best an unsatisfactory effect which does not justify the expenditure. It appears as though the amount of fertilizer must exceed a ' threshold value ' for the result to become visible. The yield curve based on increasing quantities of fertilizers has therefore an S-shaped course ".

Explaining this same trend, Lamer (1957) observes that " the production function relating fertilizers and yields depend upon the transformation of plant nutrients in fertilizers into plant food by the plant itself. If all other natural factors are maintained as constant, and attention is concentrated only on the effect of mineral nutrition, then it is noticed that the metabolic process uses, in each stage of plant growth, different proportions of the plant nutrients added to the soil in fertilizers. There is a small response on first addition because the soil retains more nutrients than the plant absorbs ; after the addition of a ' minimum effective dose ', a quick and large response follows in the plant ; this response, however, slows down as the metabolism reaches the saturation point ; beyond this point the additions of mineral nutrients have an irritating effect on the plant's plasm and can actually result in negative response in output ".

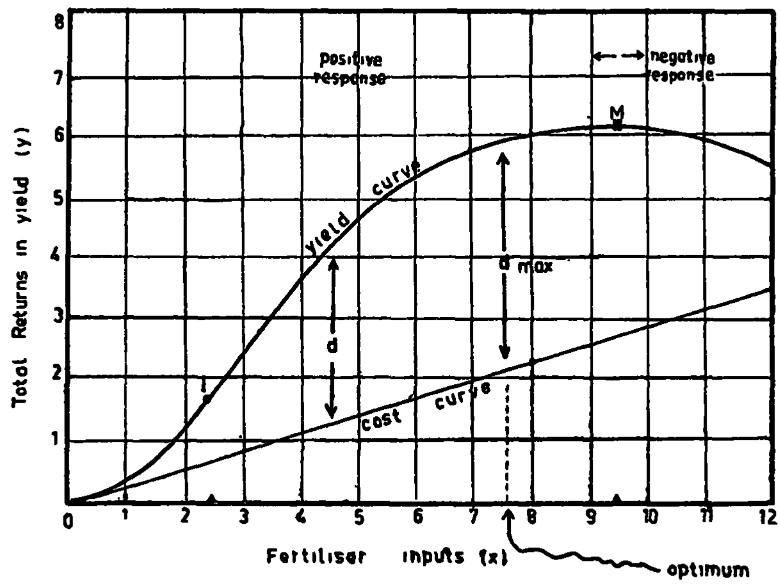
Fig. 1 gives a typical S-shaped fertilizer response curve where the soil originally (i.e. at zero level of fertilizer) can be considered almost completely void of nutrients. Near about the zero level, the response is negligible ; as the dosage is increased, there is a very marked response at first giving indications of increasing returns ; with further dosages (once the point of inflexion I is passed), the response falls away on a diminishing returns basis to reach a maximum at M ; and with very high dosages, the response is negative.

(b) Non-applicability of a standard equation in practice.

Due to the limited range of fertilizer inputs that one normally tries out in field trials and the wide variations of the plant—soil—climatic complex in which such trials are conducted, the response curve in practice does not exhibit anything called a standard pattern and therefore, for all practical purposes, the term " standard response equation " may be a misnomer. Lamer (1957) refers to this when he states that " the relationship between fertilizer inputs and crop yields in actual soil conditions can sometimes be described by a Sigmoid

Fig (1)

S - shaped Fertiliser Response Curve



I :— Point of inflexion

M :— Point of Maximum response

D :— Profit

curve, whereas at other times there is an immediate diminution of additional yields. It is necessary to determine the productivity curve in each specific case, because different conditions, species and varieties of crops, and types of fertilizers result in different responses ".

If we refer back to the generalised fertilizer response curve shown in Fig. 1, it is conceivable that, in a particular experiment, the observations may fall on different ranges of the fertilizer axis depending on the original nutrient status of the soil, the efficiency of the fertilizer used, and a host of other considerations. For instance, the ranges may be 0—3, or 3—8, or 4—10, or 6—12. Each range may call for different treatment from the point of view of fitting response curves. The pattern of response within each of these ranges as extracted from the relevant section of the general curve (Fig. 1) and also the type of empirical model that may reasonably explain each such pattern, are summarised in Chart 1, which is self explanatory.

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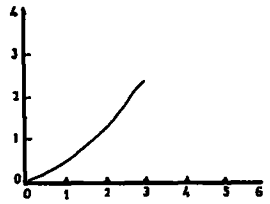
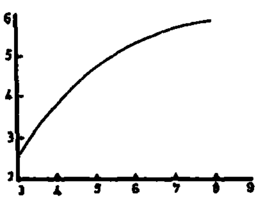
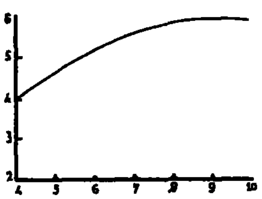
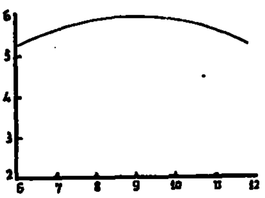
Chart I

Fertiliser Response Patterns

Range of Fertiliser (Ref. Fig. I)	Relevant Pattern of yield response	Trends of response	Suitable Empirical model
0 – 3		(1) Increasing returns	Logistic Curve
3 – 8		(1) Diminishing returns (2) No indications of depression	Cobb Douglas' or Mitscherlich's or Polynomial
4 – 10		(1) Diminishing returns (2) Slight depression	Polynomial
6 – 12		(1) Diminishing returns (2) Symmetrical before & after maximum	Polynomia: parabolic)

The first level in the range gives the original fertiliser level in the soil

From the above, it is reasonable to suggest that the idea of fitting a standard equation for fertilizer responses observed under field conditions—if such an equation exists at all—has no sound factual basis. On the other hand, such indiscriminate treatment may introduce serious divergences between observed and predicted values in some portions of the curve—a situation which is very unsatisfactory in problems of estimating optimum fertilizer dosages. Every situation, therefore generally calls for independent treatment based on its own trends as well as other scientific considerations.

(c) *Truncated nature of response curves met with in the estimation of optimum fertilizer dosages.*

While so many types of curves are applicable in determining the response patterns, the data from field experiments are such that the increasing returns part of the generalised response curve is seldom encountered unless the soil is extremely poor in nutrient content. Russel (1932) confirms this in his observation that "experiments do not always give sigmoid curves, but a simple shape showing a continuous fall in effect of the nutrient, the first increment producing the largest effect and subsequent increments less and less effect till finally no additional gain and sometimes even disadvantage ensues. Whether the sigmoid or the continuously falling curve more nearly represents the normal effect of nutrients on plant growth is not known. It is possible that the curve is usually sigmoid, but the point of inflexion is so near the origin that it is commonly missed".

However, even if there are situations in which the increasing returns portion of the curve is clearly indicated, it will be appreciated that due to limitations imposed by the experimental range of observations, such a curve very seldom extends far beyond the point of inflexion of the curve—that is the point of commencement of the diminishing returns phase. And in such a situation, the question of estimating optimum fertilizer dosages does not arise unless one is prepared to tolerate a very lengthy and doubtful extrapolation.

Lamer (1957) probably sums up the above contentions when he states that "Fertilizer experts are not interested in the part of the curve below the point of inflexion".

Therefore our problem of fitting response curves as a preliminary to estimating optimum fertilizer dosages, becomes more simplified in practice—being restricted to the regular concave (to the abscissa) type of curves following the law of diminishing returns or the "increasing—decreasing" curves as they are usually termed. In fact what we observe in practice may be only a truncated form of the S-shaped response curve—that is with the increasing returns portion absent or obscured or of no interest.

(*d*) *Variation of "Increasing—Decreasing" response curves.*

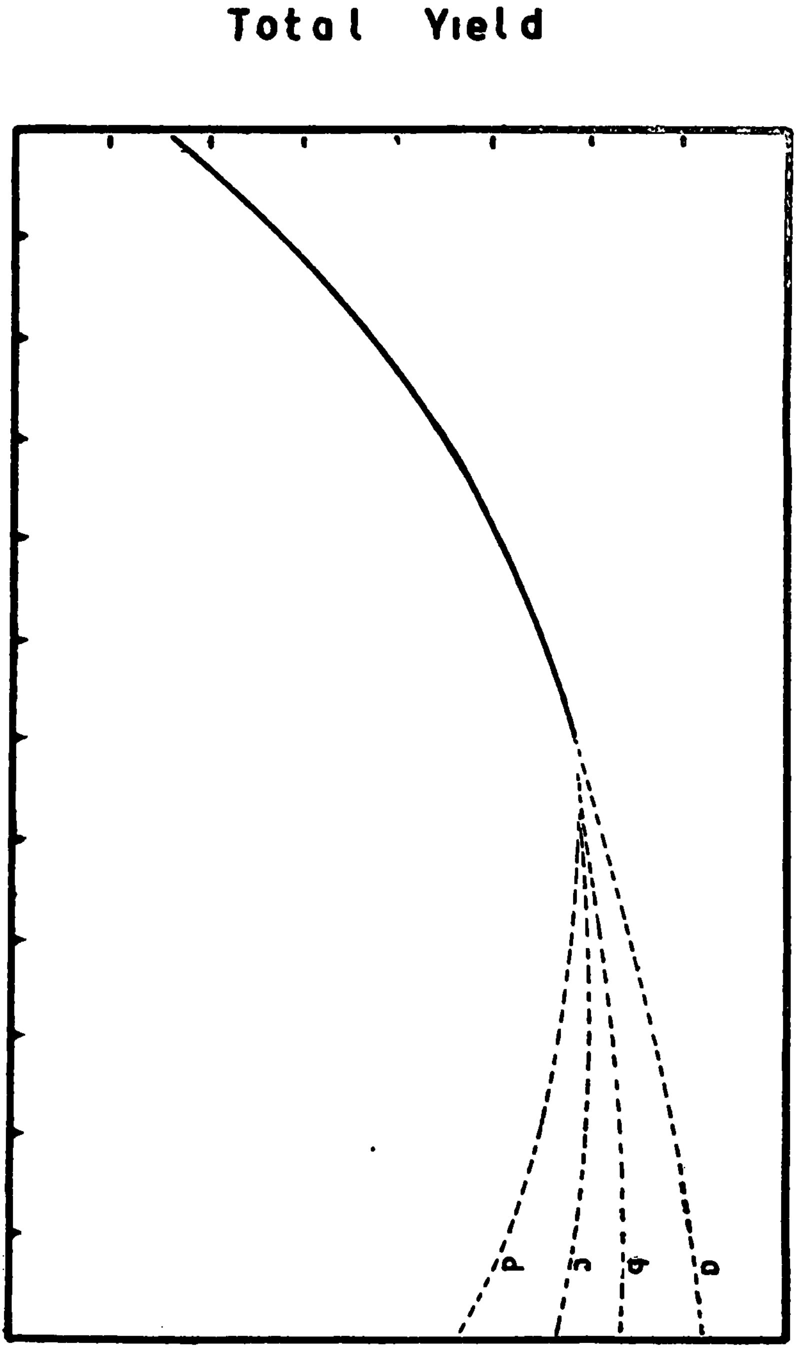
Even within the simplified increasing—decreasing pattern mentioned above, response curves can exhibit wide variations. At least four distinct types can be reasonably expected within the experimental range and these in fact may be characteristic of the soil-plant-climatic set-up under consideration.

Fig. 2 shows the "increasing decreasing" curves referred to above. The dotted extensions to the common continuous line show some distinct variations within this family of curves to be expected under field conditions. All four curves are the diminishing returns type. In (*a*) the rate of increase tends to zero but never seems to reach zero. A continuous positive response is expected, however, infinitsemal it may be. In (*b*) the rate of increase reaches zero and the curve proceeds asymptotically. Here again no depression is anticipated. In (*c*) the rate of increase falls below zero and therefore gives rise to a depression ; but the rate of fall of the gradient occurs at a decreasing rate. In (*d*) the increasing—decreasing pattern is similar to (*c*) ; but the rate of fall of the gradient is a constant. It must however, be noted, that the continuous line drawn common to all four types, is so done purely for convenience. This early section too may vary according to the above four types ; or it may even be that the portion showing the positive response follows a certain law and the portion showing the depression follows yet another law. But such refinements are beyond the scope of our present knowledge and certainly beyond the scope of the present paper where we are only interested in a single curve to fit univariate fertilizer responses.

2. RECOGNISED EMPIRICAL MODELS FOR "INCREASING—DECREASING" RESPONSE CURVES

BEFORE suggesting mathematical models for use in fitting the above type of response curves, it will be important to understand the context in which such models are suggested. Mitscherlich and a few others, when they suggested mathematical models for response curves were mainly interested in determining "constant efficiency factors", implying universality of application to some degree or other. However in problems of estimating optimum fertilizer dosages, one thinks of

Fig. 2



the response curve primarily as a convenient method of interpolation—probably with some provision also for a little extrapolation—implying thereby that no attempt whatsoever is made to establish its universal applicability. In fact Crowther and Yates (1941) when they employed an adaptation of Mitscherlich's model, did so on this same principle.

On this basis, several algebraic forms are available and have been successfully used in fitting curves to "increasing—decreasing" fertilizer responses. These include several complex higher degree curves which have given excellent fits. However the latter, although they perform well within the observed range, are generally not recommended because apart from the computational difficulties, they have, according to Anderson (1957), undesirable properties for small and large values of the level of fertilizer. He also points out that " although this undesirable feature may not be too important so long as it performs satisfactorily within the relevent range, it is important in estimating optimum fertilizer application". If then we avoid complex higher degree curves, we observe that published work indicate four specific types of functions that are popularly used in fitting fertilizer response curves. Different adaptations of these are sometimes used—for example, the adapted forms of Mitscherlich's model used by Spillman (1933) and Crowther & Yates (1941). But the main principles are those covered by these four models. Anderson (1957) in fact has used these four models in a study with respect to some special aspects of fertilizer responses. It may be noted, in passing, that Cobb-Douglas' model given herein is the Allometric equation used very often in studies on organic growth.

The models are given below :—

Model	*Equation*
(1) Mitscherlich's Model	$\hat{Y} = A - be^{-ax}$ or $\hat{Y} = A - br^{x}$ $(\log_e r = -a)$ or $\log(A - \hat{Y}) = \log b + X \log r$ (linear form)
(2) Allometric Model (Cobb-Douglas')	$Y = a + \beta x^{\alpha}$ or $\log(Y - a) = \log\beta + \alpha \log x$ (linear from)
(3) Quadratic Model ..	$Y = a + bx - cx^2$
(4) Square Root Model ..	$Y = a - bx + c\sqrt{x}$

$\hat{Y}$ is the expected yield; A is the maximum yield under ideal fertilizer conditions to be estimated from the data; x is the level of fertilizer; a, b & c in the quadratic and square root models, b, r & a of the Mitscherlich's model, and α & β of the Allometric model, are parameters to be estimated by the method of least squares; a in the Allometric model is the average yield at zero level of fertilizer; e stands for epsilon = 2.71828.

The quadratic and the square root models are polynomial types which do not presuppose any fixed theory as such. On the other hand, the Mitscherlich's and Cobb-Douglas' models—first an asymptotic type and the second an ever-increasing type—both imply a distinct expectation with respect to the pattern of fertilizer response. The latter equations in their simplified linear forms may be defined as follows:—

(*a*) Mitscherlich's equation:—

The logarithm of the decrement in yield from the maximum yield is linearly (negatively) correlated with the fertilizer level.

(*b*) Cobb-Douglas' equation:—

The logarithm of the increment in yield over the yield at zero level of fertilizer is linearly (positively) correlated with the logarithm of the fertilizer level.

3. A COMPARISON OF THE EMPIRICAL MODELS

(*a*) *Mathematical considerations.*

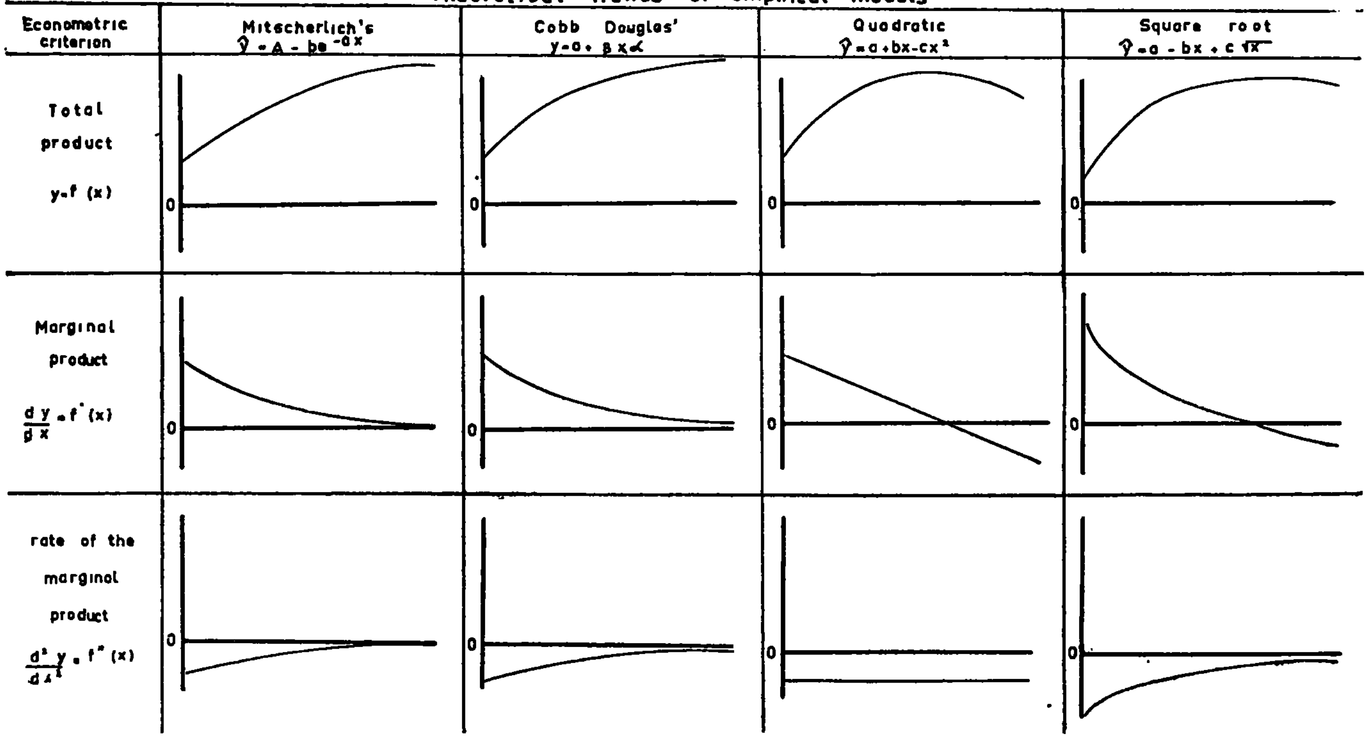
In studies on dynamics of production with respect to any effective "input" (yield versus fertilizer in this instance), Econometricians conventionally direct their attention on three specific aspects of the production curve—namely;

(1) Curve of total product (primary curve).
(2) Curve of marginal product (first derived curve).
(3) Curve of the rate of the marginal product (second derived curve).

The curve of the total product (y), in the present context, is the total yield curve with respect to the increasing levels of fertilizer (x), The curve of the marginal product is the curve of the gradient (dy/dx) of the yield curve, and shows the potential productive capacity of additional fertilizers. The rate of the marginal product is given by

Chart II

Theoretical Trends of empirical models



the curve of the gradient of the gradient (d^2y/dx^2) of the yield curve and shows the tempo of the potential productive capacity of additional fertilizers.

The four mathematical models suggested earlier will be studied herein through these three specific criteria and their trends will first be examined on the basis of mathematical considerations.

Chart II, gives the general trends of the primary and the first and second derived curves (namely of y, dy/dx and d^2y/dx^2) with respect to each of the four models. These, as explained above, correspond to the curves of total product, marginal product and the rate of marginal product respectively.

The curves (though not drawn to scale) indicate the following broad patterns in respect of each of the three econometric criteria.

(1) Total product curve (y).

All equations show a positive response at the outset. But the Mitscherlich's model reaches a definite maximum and continues at that level ; Cobb-Douglas' model tends to, although it does not really reach a finite maximum ; and in the polynomial models—quadratic and square root—the response reaches a certain maximum and is then followed by a negative response.

(2) Marginal product curve (dy/dx).

From the point of view of the marginal product, all four models show diminishing returns by their downward trends. However, the Mitscherlich's and Cobb-Douglas' models set a limit of zero to the marginal product (the latter does not reach zero), whereas the polynomial models can cater to a negative marginal product too.

(3) Rate of marginal product (d^2y/dx^2).

The rate of the marginal product is negative in all the curves and this is necessarily implied in all forms of responses following the law of diminishing returns. The rate of this decrease of the marginal product itself decreases in the case of Mitscherlich's, Cobb-Douglas' and the square root models—this again being in keeping with most forms of biological responses. However the quadratic model is different, in that its rate of the marginal product is a constant.

(b) *Theoretical trends of models vis a vis trends of fertilizer responses.*

The efficiency of a mathematical model used to determine the law behind a given set of observed data in respect of some problem, has to be guaged by the flexibility of the model—that is its adaptability to the whole range of variation expected in the problem. Therefore the theoretical background of the trends set out above in rspect of the four models, has to be examined in the light of certain reasonably established basic trends of "increasing—decreasing" fertilizer responses, which may be summarised as follows :—

(1) A positive response in the total product giving rise to a maximum later. Still later a negative response may or may not be observed.

(2) Decreasing positive marginal returns tending to zero. This may or may not be followed by negative marginal returns.

(3) Decreasing rate of fall away of marginal returns in most cases.

From the above it follows that (1) the polynomial models are superior to the other models in that the former can accommodate even depressing yields, and (2) from the point of view of the pattern of the marginal product, Mitscherlich's, Cobb-Douglas' and the square root models are superior to the quadratic model, because the latter suffers from the restriction of a constant rate of fall away of the marginal product.

Therefore one may reasonably conclude that, on a broad basis, the square root model is the most flexible type. However it is not contended that in a specific set of circumstances, any one of the other models would fit better and may even be more meaningful.

(c) *Comparisons on the basis of applications on experimental data.*

The above four models have been fitted on fertilizer response data based on four or more levels of nutrient application in respect of certain experiments conducted at the Coconut Research Institute of Ceylon. The data selected were only those that showed statistical significance in the Analysis of Variance.

The Mitscherlich's and Cobb-Douglas' models have been fitted on the basis of linear equations in respect of these models given earlier, transforming one or both variables as the case may be. In the case of Mitscherlich's model, the maximum response (A) has been estimated by Stevens' (1951) method of successive approximation reported by Anderson (1957). The other two equations have been fitted by the usual method of Multiple Regression.

A comparative index of the goodness of fit in respect of each model is calculated by

$$R \text{ (or } r) = \sqrt{\frac{\text{Sums of squares due to regression}}{\text{Total sums of squares of y}}}$$

Table I gives (1) the source of data (2) the variety of plant (3) the nutrient applied (4) number of levels including zero level and (5) index of the goodness of fit R or r.

The Mitscherlich's and Cobb-Douglas' models have not been fitted on data which showed any depression in yield at higher levels of fertilizer—being obviously unsuitable.

The observed and expected yields are not given due to lack of space.

The comparative merits of the four models explained earlier on the basis of mathematical considerations are amply confirmed by these applied results.

In general, polynomial models are better than Mitscherlich's and Cobb-Douglas' models ; and the square root model is more often than not superior to the quadratic model. Further the fact that in some specific cases, any one of the four models may suit better is also confirmed. It may also be noted that even in those few instances where the quadratic model is superior to the square root model, the extent of the superiority is not so large as to discourage one from recommending the square root model as a generally applicable model for " increasing—decreasing " curves.

DISCUSSION

The selection of a suitable mathematical model for fitting fertilizer response curves, calls for considerable judgement both from the point of view of the mathematical trends of the models available as alternatives as well as the trends observed in the data and also expected from scientific and other *ad hoc* considerations.

However, in practice, the need for that much of attention, does not seem to be seriously felt. Some work on the basis that " the proof of the pudding is in the eating of it ", and therefore select the model that gives the least residual error ; others stolidly lay their faith on a model with some popular law behind it—such as the Mitcherlich's—and attribute any deviations therefrom, however regular they may be, to experimental errors ; quite a few also prefer to use the common quadratic model in all instances regardless of its implications.

TABLE I

A Comparison of Mathematical Models on the Basis of Application on Experimental Data

Source of Data	*Variety*	*Nutri-ent*	*No. of Levels*	R (or r) = $\sqrt{\frac{\text{S. S. Due to Regression}}{\text{S. S. Due to } y}}$			
				Mitsche Rlich	*Cobb-Doug.*	*Quadra-Tic.*	*Square Root*
C 173 First Harvest ..	P. C.	K	4	0.9964	0.9816	0.9999	0.9961
C 194 First Harvest ..	P. C.	K	4	0.9892	0.9630	0.9774	0.9996
C 159 First Harvest ..	P. C.	P	4	0.9487	0.8090	0.9578	0.9861
C 189 First Harvest ..	P. C.	N	4	0.9734	0.9925	0.9887	0.9986
C 189 Thinings ..	P. C.	N	4	0.9849	0.9838	0.9858	0.9996
C 180 Second Harvest	P. C.	N	4	0.9989	0.9999	0.9985	0.9999
C 194 First Harvest ..	P. C.	P	4	0.9850	0.9999	0.9917	0.9999
C 173 First Harvest ..	P. C.	N	4	0.9825	0.9593	0.9999	0.9928
C 180 First Harvest ..	P. C.	K	4	0.9727	0.9807	0.9723	0.9979
ESTATE A (R.R.I) ..	RUBBER	N	4	0.9845	0.9724	0.9656	0.9947
ESTATE A (R.R.I) ..	RUBBER	K	4	0.9991	0.9850	0.9989	0.9983
C 174 Third Harvest ..	M. S.	S	5	—	—	0.8066	0.9202
C 160 Thinings ..	P. C.	P	4	—	—	0.8238	0.9211
C 173 Third Harvest..	P. C.	N	4	—	—	0.7711	0.9209
C 189 Second Harvest	P. C.	N	4	—	—	0.9991	0.9766
C 159 Thinings ..	P. C.	P	4	—	—	0.9907	0.9972
C 194 Second Harvest	P. C.	K	4	—	—	0.9999	0.9853
C 160 First Harvest ..	P. C.	P	4	—	—	0.9605	0.9982
C 173 Thinings ..	P. C.	N	4	—	—	0.9286	0.9958

Note :—(1) Experiments on Paspalum Commersonii (P.C.) and Madicago Sativa (M.S.) have been conducted in a "Phytosolarium" at the Coconut Research Institute of Ceylon.

(2) R. R. I. : Rubber Research Institute of Ceylon.

The last approach is, obviously, not quite sound, although there may be instances where the quadratic model may fit better and validly so. The first two approaches, on the other hand, can under certain circumstances be quite justifiable. Generally the best attitude would be to consider these as alternative approaches in different situations.

In general, when one has nothing to go by—such as some ancilliary information regarding expected broad trends in the yield pattern in the particular set of circumstances—and when the number of levels of fertilizer tested is too few to decipher any regular or irregular tendencies in the observed data, it is recommended that one should resort to the more flexible type of mathematical model. On the basis of the present study, such a flexible model is provided by the square root equation.

If the data were based on several replications, which will bring the observed values close to the expected values and/or when there are (say) 5 or more levels of fertilizer, then the model that gives the better fit is to be preferred.

When the observed data show any depression in yield at higher levels of fertilizer, one has the option only between the square root and the quadratic—the other two models being obviously unsuitable, unless the experimenter is confident that the depression in yield is due to some other extraneous causes. The decision again rests on the better fit, provided a sufficient number of levels have being tested. If, however, the number of levels is too few (say, not exceeding four), and if one hopes to extraplote a bit, it is preferable to use the square root form. At any rate the quadratic form is to be avoided unless one is pretty certain that the pattern of depression after the point of maximum response runs symmetrically to the pattern of response before. If as mentioned earlier, any depression in yield or for that matter any other value, is known to be the obvious result of some extraneous factor independent of the fertilizer, then either this value should be dropped from consideration or some correction may be made (say) using some concomitant variable.

For the sake of completeness, it will be pertinent to give a few comments on the fitting of curves to data based on a minimum of three fertilizer levels—usually zero, one and two. It is undoubtedly desirable to have at least one fertilizer level more than the number of constants in the equation fitted, because otherwise the residual error is zero and as a result it is not possible to get any idea of the comparative goodness of fit of alternative models. But there are numerous situation where we have only three levels and we still need to get some idea of the optimum fertilizer dosage or the yield curve. In such situations there is nothing biologically objectionable in fitting a curve to 3 points, provided one is conscious of the limitations of such curves and also takes some care with respect to the model to be used.

In passing, it will be of interest to note that the square root model, can be applied successfully to a wide variety of relationships showing diminishing returns. The author (1962) has found it superior to the quadratic form in problems of correcting (through multiple covariance analysis) for the curvilinear effect of the stand of palms on the plot yields in coconut and also for the curvilinear effect of the bearing age of coconut palms on their yields. The reduction in the "error" through this approach has been considerable.

SUMMARY

Some popular notions regarding the generalised fertilizer response pattern are reviewed. It is explained that under field conditions, the response curve, relevant to problems of estimating optimum fertilizer dosages, exhibits itself only as a restricted or truncated form of the general s-shaped response curve—namely the "increasing—decreasing" or concave to the abscissae type of curve, which itself may take any one of four specific forms or their intermediates.

Four popular mathematical models used in fitting such curves are given. These are compared on the basis of mathematical considerations through three accepted econometric criteria applicable to production functions and also on the basis of application on experimental data. Both on the basis of mathematical considerations and on the basis of results of application on experimental data, it is noted that (1) the polynomial models are more generally applicable (2) among the polynomial models, the square root model is the more adaptable, and (3) there are situations in which any one of the four models may be better.

Certain hints of guidance in the selection of a model for fitting fertilizer response curves are given.

Taking these facts into consideration, if one is confronted with the question of deciding on a universally applicable or a standard model for fitting curves to fertilizer responses or for that matter any responses showing diminishing returns, it would be reasonable to suggest that the square root model is relatively the most acceptable.

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